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Transition radiation from an anisotropic dielectric layer with particular reference to the biological membrane†

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Abstract. This paper considers the problem of transition radiation, at oblique particle incidence, from a homogeneous uniaxial dielectric layer imbedded between two semi-infinite homogeneous isotropic dielectric media. The vector wave problem is transformed into two independent boundary-value problems for a pair of scalar potential functions. Exact expressions for the radiation field are derived.

1. Introduction

An area of intensive research in contemporary biophysics is the study of the structure and function of the biological membrane. Of particular interest is the mechanism of ionic transport across the membrane—a process so important to the vital functioning of organic matter. In the formulation of a theory for such a process, it is desirable to have at hand some knowledge of the thickness and dielectric properties of the membrane. Previous investigations, based on capacitance and reflectivity measurements, have concluded that the membrane is about 50 Å to 100 Å thick, and that its dielectric constant is about 2·5 times the free-space value. There are also indications that the membrane is actually anisotropic, the dielectric constant being slightly smaller along the normal to the membrane surface (Ohki 1969).

In this work we consider a new technique for measuring the properties of the membrane, which may serve to check or improve on the previous results, by probing the membrane with high-speed charged particles of great penetrating power. As was shown by Ginzburg and Frank (1946), a charged particle, crossing the interface between two different media, must emit a radiation, called transition radiation, which is characteristic of the electromagnetic properties of the media. By observing the emitted radiation we may infer the properties of the media along the trajectory of the particle. Much interest in the study of transition radiation has recently been developed in connection with the optics of thin films. However, despite the fact that the volume of literature in this field is extensive and steadily increasing, as is evidenced by the bibliography of a review article of Bass and Yakovenko (1965), there does not seem to be a published calculation whose results are immediately applicable to the membrane problem. We therefore take up the task and formulate the problem as follows.

We consider as a model of the biological membrane a homogeneous uniaxial dielectric sheet of thickness $2a$ and infinite extent, lying between the surfaces $z = \pm a$. We call it medium 2. Its dielectric tensor is given by

$$\epsilon_2 = \begin{pmatrix} \epsilon' & 0 & 0 \\ 0 & \epsilon' & 0 \\ 0 & 0 & \epsilon'' \end{pmatrix}. \tag{1.1}$$

The half-space $z < -a$ is occupied by a homogeneous isotropic medium 1 of dielectric constant ϵ_1; the half-space $z > a$ is occupied by a different homogeneous isotropic medium 3 of dielectric constant ϵ_3. All three media are taken to be non-magnetic, so that their permeabilities assume the free-space value μ_0. A point particle of charge e is incident from medium 1 on the interfaces $z = \pm a$. Since the energy loss due to the emission of transition radiation is only a very small fraction ($< 1\%$) of the particle's kinetic energy,

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we shall assume that the velocity of the particle is constant. Without further loss of generality we write down the charge and current densities associated with the particle's motion as follows:

$$\begin{aligned} \rho(\boldsymbol{r}, t) &= e\delta^3(\boldsymbol{r}-\boldsymbol{v}t) \\ \boldsymbol{J}(\boldsymbol{r}, t) &= \rho(\boldsymbol{r}, t)\boldsymbol{v} \\ \boldsymbol{v} &= v_x\boldsymbol{e}_x + v_z\boldsymbol{e}_z \end{aligned} \tag{1.2}$$

where v_x and v_z are constants. The aim of this paper is to calculate the transition radiation in medium 1. As formulated in this way, our problem is sufficiently general to include several previous results as special cases.

In § 2 the field of a point charge moving uniformly in a homogeneous uniaxial medium is derived. In § 3 we introduce two scalar functions, the Bromwich potentials, to reduce the vector wave problem into two scalar ones. The resulting boundary-value problems are solved in § 4, and the nature of the solutions is examined in § 5. Throughout this work we use the m.k.s. system of units.

2. The primary wave

Maxwell's equations in a general medium have the form

$$\begin{aligned} \nabla . \boldsymbol{D}(\boldsymbol{r}, t) &= \rho(\boldsymbol{r}, t), & \nabla\times \boldsymbol{E}(\boldsymbol{r}, t) &= -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t) \\ \nabla . \boldsymbol{B}(\boldsymbol{r}, t) &= 0, & \nabla\times \boldsymbol{H}(\boldsymbol{r}, t) &= \frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r}, t)+\boldsymbol{J}(\boldsymbol{r}, t). \end{aligned} \tag{2.1}$$

Taking the Fourier transform in the time variable, we obtain

$$\begin{aligned} \nabla . \boldsymbol{D}(\boldsymbol{r}, \omega) &= \rho(\boldsymbol{r}, \omega), & \nabla\times \boldsymbol{E}(\boldsymbol{r}, \omega) &= \mathrm{i}\omega\boldsymbol{B}(\boldsymbol{r}, \omega) \\ \nabla . \boldsymbol{B}(\boldsymbol{r}, \omega) &= 0, & \nabla\times \boldsymbol{H}(\boldsymbol{r}, \omega) &= -\mathrm{i}\omega\boldsymbol{D}(\boldsymbol{r}, \omega)+\boldsymbol{J}(\boldsymbol{r}, \omega) \end{aligned} \tag{2.2}$$

where

$$\boldsymbol{J}(\boldsymbol{r}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{J}(\boldsymbol{r}, t), \qquad \text{etc.} \tag{2.3}$$

We introduce the constitutive relations

$$\boldsymbol{D}(\boldsymbol{r}, \omega) = \boldsymbol{\epsilon}(\omega) . \boldsymbol{E}(\boldsymbol{r}, \omega), \qquad \boldsymbol{B}(\boldsymbol{r}, \omega) = \mu_0\boldsymbol{H}(\boldsymbol{r}, \omega) \tag{2.4}$$

where $\boldsymbol{\epsilon}(\omega)$ is the dielectric tensor, which, in general, is frequency dependent. Then we have from (2.2) and (2.4)

$$\nabla\nabla . \boldsymbol{E} - \nabla^2\boldsymbol{E} - \omega^2\mu_0\boldsymbol{\epsilon} . \boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{J}. \tag{2.5}$$

The primary field is the particular solution of (2.5), with $\boldsymbol{J}$ given by (1.2), which satisfies the radiation condition. We denote it by $\boldsymbol{E}^0$.

We express $\boldsymbol{J}(\boldsymbol{r}, \omega)$ in the form of a double Fourier integral:

$$\boldsymbol{J}(\boldsymbol{r}, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta \exp\{\mathrm{i}(\alpha x+\beta y)\}\boldsymbol{J}(\alpha, \beta, z, \omega). \tag{2.6}$$

Then

$$\begin{aligned} \boldsymbol{J}(\alpha, \beta, z, \omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \exp\{-\mathrm{i}(\alpha x+\beta y-\omega t)\}\boldsymbol{J}(\boldsymbol{r}, t) \\ &= \frac{\mathrm{e}}{v_z}\mathrm{e}^{\mathrm{i}\gamma z}\,\boldsymbol{v} \end{aligned} \tag{2.7}$$

where

$$\gamma = \frac{1}{v_z}(\omega - \alpha v_x). \tag{2.8}$$

Defining

$$\boldsymbol{J}(\alpha, \beta, z, \omega) = \mathrm{e}^{\mathrm{i}\gamma z}\boldsymbol{J}(\alpha, \beta, \omega) \tag{2.9}$$

$$\boldsymbol{J}(\alpha, \beta, \omega) = \frac{e}{v_z}\boldsymbol{v} \tag{2.10}$$

$$\boldsymbol{k} = (\alpha, \beta, \gamma) \tag{2.11}$$

we can re-express (2.6) as

$$\boldsymbol{J}(\boldsymbol{r}, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta \exp(\mathrm{i}\boldsymbol{k} \,.\, \boldsymbol{r})\boldsymbol{J}(\alpha, \beta, \omega). \tag{2.12}$$

Similarly, we define

$$\boldsymbol{E}^0(\boldsymbol{r}, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta \exp(\mathrm{i}\boldsymbol{k} \,.\, \boldsymbol{r})\boldsymbol{E}^0(\alpha, \beta, \omega). \tag{2.13}$$

Then (2.5) reduces to

$$\Lambda(\boldsymbol{k}) \,.\, \boldsymbol{E}^0(\alpha, \beta, \omega) = \mathrm{i}\omega\mu_0\boldsymbol{J}(\alpha, \beta, \omega). \tag{2.14}$$

Here $\Lambda(\boldsymbol{k})$ is a dyadic operator given by

$$\Lambda(\boldsymbol{k}) = -\boldsymbol{k}\boldsymbol{k} + k^2 I - \omega^2\mu_0\epsilon \tag{2.15}$$

with I the unit dyadic. Using the expression for ϵ in (1.1), we write out (2.15) in the form of a matrix:

$$\Lambda(\boldsymbol{k}) = \begin{pmatrix} k^2 - \alpha^2 - \omega^2\mu_0\epsilon' & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & k^2 - \beta^2 - \omega^2\mu_0\epsilon' & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & k^2 - \gamma^2 - \omega^2\mu_0\epsilon'' \end{pmatrix}. \tag{2.16}$$

The solution of (2.14) is then

$$\boldsymbol{E}^0(\alpha, \beta, \omega) = \Lambda^{-1}(\boldsymbol{k}) \,.\, \mathrm{i}\omega\mu_0\boldsymbol{J}(\alpha, \beta, \omega) \tag{2.17}$$

$\Lambda^{-1}(\boldsymbol{k})$ being the inverse of $\Lambda(\boldsymbol{k})$. By a well-known result in matrix theory we have

$$\Lambda^{-1}(\boldsymbol{k}) = \frac{1}{\det \Lambda(\boldsymbol{k})}\tilde{\Lambda}(\boldsymbol{k}) \tag{2.18}$$

where $\tilde{\Lambda}(\boldsymbol{k})$ is the adjugate matrix of $\Lambda(\boldsymbol{k})$. $\tilde{\Lambda}(\boldsymbol{k})$ is symmetric and its elements are

$$\begin{aligned}
\tilde{\Lambda}_{11} &= (k^2 - \beta^2 - \omega^2\mu_0\epsilon')(k^2 - \gamma^2 - \omega^2\mu_0\epsilon'') - \beta^2\gamma^2 \\
\tilde{\Lambda}_{22} &= (k^2 - \alpha^2 - \omega^2\mu_0\epsilon')(k^2 - \gamma^2 - \omega^2\mu_0\epsilon'') - \alpha^2\gamma^2 \\
\tilde{\Lambda}_{33} &= (k^2 - \alpha^2 - \omega^2\mu_0\epsilon')(k^2 - \beta^2 - \omega^2\mu_0\epsilon') - \alpha^2\beta^2 \\
\tilde{\Lambda}_{12} &= \alpha\beta(k^2 - \omega^2\mu_0\epsilon'') \\
\tilde{\Lambda}_{13} &= \alpha\gamma(k^2 - \omega^2\mu_0\epsilon') \\
\tilde{\Lambda}_{23} &= \beta\gamma(k^2 - \omega^2\mu_0\epsilon').
\end{aligned} \tag{2.19}$$

Moreover, we have

$$\det \Lambda(\boldsymbol{k}) = -\omega^2\mu_0(k^2 - \omega^2\mu_0\epsilon')\{\epsilon'(\alpha^2 + \beta^2) + \epsilon''\gamma^2 - \omega^2\mu_0\epsilon'\epsilon''\}. \tag{2.20}$$

In the special case of an isotropic medium $\epsilon' = \epsilon'' = \epsilon$, and (2.17) reduces to

$$\boldsymbol{E}^0(\alpha, \beta, \omega) = \frac{1}{k^2 - \omega^2\mu_0\epsilon}\left(I - \frac{\boldsymbol{kk}}{\omega^2\mu_0\epsilon}\right) . \, \mathrm{i}\omega\mu_0 \boldsymbol{J}(\alpha, \beta, \omega). \tag{2.21}$$

Thus we have calculated the primary field due to the motion of the charged particle. In particular, for the z component we have

$$E_z{}^0(\boldsymbol{r}, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{d}\beta \exp(\mathrm{i}\boldsymbol{k} \, . \, \boldsymbol{r}) \frac{-ie}{\omega v_z} \frac{\alpha\gamma v_x + (\gamma^2 - \omega^2\mu_0\epsilon')v_z}{\epsilon'(\alpha^2+\beta^2) + \epsilon''\gamma^2 - \omega^2\mu_0\epsilon'\epsilon''}. \tag{2.22}$$

By (2.2), it follows from (2.17) that

$$B_z{}^0(\boldsymbol{r}, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{d}\beta \exp(\mathrm{i}\boldsymbol{k} \, . \, \boldsymbol{r}) \frac{-\mathrm{i}\mu_0 e}{v_z} \frac{\beta v_x}{k^2 - \omega^2\mu_0\epsilon'}. \tag{2.23}$$

The integrals (2.22) and (2.23) will satisfy the radiation condition if we formally give ω a small positive imaginary part.

The total field in each of the three media is a superposition of the primary field and a solution of the homogeneous Maxwell equations. The latter represents the transition radiation, and is uniquely determined by the requirement that the total field satisfies the following boundary conditions at each of the interfaces:

$$\begin{aligned} D_{\mathrm{n}+}^{\mathrm{tot}} &= D_{\mathrm{n}-}^{\mathrm{tot}}, & E_{\mathrm{t}+}^{\mathrm{tot}} &= E_{\mathrm{t}-}^{\mathrm{tot}} \\ B_{\mathrm{n}+}^{\mathrm{tot}} &= B_{\mathrm{n}-}^{\mathrm{tot}}, & H_{\mathrm{t}+}^{\mathrm{tot}} &= H_{\mathrm{t}-}^{\mathrm{tot}} \end{aligned} \tag{2.24}$$

where n and t refer to the normal and tangential components respectively, and + and − refer to the two sides of the interface.

3. The Bromwich potentials

In a source-free region of space we can separate an electromagnetic wave into a transverse electric and a transverse magnetic component relative to the z direction. This decomposition is accomplished with the help of two scalar potentials U and V, which are generalizations of the ones used by Bromwich (1919) for an isotropic medium. We shall refer to them as the Bromwich potentials.

Consider first the transverse electric wave. We introduce a scalar function $U(\boldsymbol{r}, \omega)$ such that

$$\boldsymbol{E} = \mathrm{i}\omega\nabla \times (U\boldsymbol{e}_z). \tag{3.1}$$

By construction $\boldsymbol{E}$ is transverse to the z direction. Then

$$\boldsymbol{D} = \epsilon \, . \, \boldsymbol{E} = \mathrm{i}\omega\epsilon'\nabla \times (U\boldsymbol{e}_z) \tag{3.2}$$

and

$$\boldsymbol{B} = \frac{1}{\mathrm{i}\omega}\nabla \times \boldsymbol{E} = \nabla \times \nabla \times (U\boldsymbol{e}_z). \tag{3.3}$$

$\boldsymbol{D}$ and $\boldsymbol{B}$ are clearly divergenceless. The remaining Maxwell equation

$$\nabla \times \boldsymbol{B} = -\mathrm{i}\omega\mu_0\boldsymbol{D} \tag{3.4}$$

implies that

$$\nabla \times \{\nabla\nabla . (U\boldsymbol{e}_z) - \nabla^2 U\boldsymbol{e}_z - \omega^2\mu_0\epsilon' U\boldsymbol{e}_z\} = 0. \tag{3.5}$$

Since the curl of a gradient vanishes identically, (3.5) is satisfied if U is a solution of the scalar wave equation

$$(\nabla^2 + \omega^2\mu_0\epsilon')U = 0. \tag{3.6}$$

Hence a scalar wave function satisfying (3.6) generates a transverse electric wave given by (3.1) and (3.3). Writing these out in component form, we have

$$\begin{aligned} E_x &= \mathrm{i}\omega\frac{\partial U}{\partial y}, & B_x &= \frac{\partial^2 U}{\partial x\partial z} \\ E_y &= -\mathrm{i}\omega\frac{\partial U}{\partial x}, & B_y &= \frac{\partial^2 U}{\partial y\partial z} \\ E_z &= 0, & B_z &= -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)U \\ & & &= \left(\frac{\partial^2}{\partial z^2}+\omega^2\mu_0\epsilon'\right)U. \end{aligned} \tag{3.7}$$

We see that U can be determined from B_z.

Similarly we generate a transverse magnetic wave by introducing a scalar function $V(\boldsymbol{r}, \omega)$ such that

$$\boldsymbol{B} = -\mathrm{i}\omega\mu_0\epsilon'\nabla\times(V\boldsymbol{e}_z). \tag{3.8}$$

By construction $\boldsymbol{B}$ is transverse to the z direction. Then

$$\boldsymbol{D} = -\frac{1}{\mathrm{i}\omega\mu_0}\nabla\times\boldsymbol{B} \tag{3.9}$$

and both $\boldsymbol{B}$ and $\boldsymbol{D}$ are clearly divergenceless. Also

$$\boldsymbol{E} = \epsilon^{-1}\,.\,\boldsymbol{D} = \epsilon'\epsilon^{-1}\,.\,\nabla\times\nabla\times(V\boldsymbol{e}_z) \tag{3.10}$$

where ϵ^{-1} is the inverse of ϵ:

$$\epsilon^{-1} = \begin{pmatrix} \frac{1}{\epsilon'} & 0 & 0 \\ 0 & \frac{1}{\epsilon'} & 0 \\ 0 & 0 & \frac{1}{\epsilon''} \end{pmatrix}. \tag{3.11}$$

To satisfy the remaining Maxwell equation

$$\nabla\times\boldsymbol{E} = \mathrm{i}\omega\boldsymbol{B} \tag{3.12}$$

we have

$$\nabla\times\{\epsilon'\epsilon^{-1}\,.\,\nabla\nabla.(V\boldsymbol{e}_z)-\epsilon'\epsilon^{-1}\,.\,\boldsymbol{e}_z\nabla^2V-\omega^2\mu_0\epsilon'V\boldsymbol{e}_z\} = 0. \tag{3.13}$$

Since

$$\epsilon^{-1}\,.\,\nabla = \frac{1}{\epsilon'}\nabla-\left(\frac{1}{\epsilon'}-\frac{1}{\epsilon''}\right)\frac{\partial}{\partial z}\boldsymbol{e}_z \tag{3.14}$$

(2.13) becomes

$$\nabla\times\left(\nabla\frac{\partial V}{\partial z}-\frac{\epsilon''-\epsilon'}{\epsilon''}\frac{\partial^2 V}{\partial z^2}\boldsymbol{e}_z-\frac{\epsilon'}{\epsilon''}\nabla^2V\boldsymbol{e}_z-\omega^2\mu_0\epsilon'V\boldsymbol{e}_z\right) = 0. \tag{3.15}$$

This equation is satisfied if V is a solution of the scalar wave equation:

$$\left\{\epsilon'\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\epsilon''\frac{\partial^2}{\partial z^2}+\omega^2\mu_0\epsilon'\epsilon''\right\}V = 0. \tag{3.16}$$

Writing out (3.8) and (3.10) in component form, we have

$$
\begin{aligned}
E_x &= \frac{\partial^2 V}{\partial x \partial z} & B_x &= -i\omega\mu_0\epsilon' \frac{\partial V}{\partial y} \\
E_y &= \frac{\partial^2 V}{\partial y \partial z} & B_y &= i\omega\mu_0\epsilon' \frac{\partial V}{\partial x} \\
E_z &= -\frac{\epsilon'}{\epsilon''}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V & B_z &= 0. \\
&= \left(\frac{\partial^2}{\partial z^2}+\omega^2\mu_0\epsilon'\right)V
\end{aligned}
\tag{3.17}
$$

We see that V can be determined from E_z.

The total field is the sum of these two components:

$$
\begin{aligned}
\boldsymbol{E}^{\text{tot}} &= i\omega\nabla\times(U^{\text{tot}}\boldsymbol{e}_z)+\epsilon'\epsilon^{-1}\,.\,\nabla\times\nabla\times(V^{\text{tot}}\boldsymbol{e}_z) \\
\boldsymbol{B}^{\text{tot}} &= \nabla\times\nabla\times(U^{\text{tot}}\boldsymbol{e}_z)-i\omega\mu_0\epsilon'\nabla\times(V^{\text{tot}}\boldsymbol{e}_z).
\end{aligned}
\tag{3.18}
$$

Moreover, both U^{tot} and V^{tot} can be expressed as the sum of two parts contributed from the primary and the secondary waves:

$$
\begin{aligned}
U^{\text{tot}} &= U^0+U \\
V^{\text{tot}} &= V^0+V.
\end{aligned}
\tag{3.19}
$$

Since $\boldsymbol{E}^0$ and $\boldsymbol{B}^0$ satisfy the homogeneous Maxwell equations outside the source, we can determine U^0 and V^0 from $B_z{}^0$ and $E_z{}^0$ respectively in this region. By (2.22) and (2.23) we obtain

$$
U^0(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha\, d\beta \exp(i\boldsymbol{k}\,.\,\boldsymbol{r})\frac{i\mu_0 e}{v_z}\,\frac{\beta v_x}{k^2-\omega^2\mu_0\epsilon'}\,\frac{1}{\gamma^2-\omega^2\mu_0\epsilon'} \tag{3.20}
$$

and

$$
V^0(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha\, d\beta \exp(i\boldsymbol{k}\,.\,\boldsymbol{r})\frac{ie}{\omega v_z}\,\frac{\alpha\gamma v_x+(\gamma^2-\omega^2\mu_0\epsilon')v_z}{\epsilon'(\alpha^2+\beta^2)+\epsilon''\gamma^2-\omega^2\mu_0\epsilon'\epsilon''} \times\frac{1}{\gamma^2-\omega^2\mu_0\epsilon'}. \tag{3.21}
$$

We note that U^0 vanishes in the case of normal incidence ($v_x = 0$).

The Bromwich potentials U and V of the secondary wave are solutions of the homogeneous scalar wave equations (3.6) and (3.16), respectively. We represent them in the form of double Fourier integrals by analogy with (3.20) and (3.21):

$$
U(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha\, d\beta \exp\{i(\alpha x+\beta y+pz)\}U(\alpha,\beta,\omega) \tag{3.22}
$$

and

$$
V(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha\, d\beta \exp\{i(\alpha x+\beta y+qz)\}V(\alpha,\beta,\omega). \tag{3.23}
$$

Substituting (3.22) and (3.23) into (3.6) and (3.16), we obtain the following dispersion relations for the determination of p and q:

$$
\alpha^2+\beta^2+p^2-\omega^2\mu_0\epsilon' = 0 \tag{3.24}
$$

and

$$
\frac{\epsilon'}{\epsilon''}(\alpha^2+\beta^2)+q^2-\omega^2\mu_0\epsilon' = 0. \tag{3.25}
$$

These two equations are recognized to be the dispersion relations for the ordinary and the extraordinary waves, respectively, in a uniaxial crystal. For an isotropic medium they degenerate into one, and p and q are the same.

The amplitude functions $U(\alpha, \beta, \omega)$ and $V(\alpha, \beta, \omega)$ of the secondary wave are determined by the boundary conditions. Translated in terms of the Bromwich potentials, the conditions (2.24) imply the continuity of the quantities

$$U^{\text{tot}}, \qquad \frac{\partial}{\partial z} U^{\text{tot}}, \qquad \epsilon' V^{\text{tot}}, \qquad \frac{\partial}{\partial z} V^{\text{tot}}$$

across the interfaces $z = \pm a$, as can be easily seen from (3.7) and (3.17).

4. The secondary wave

Before applying the boundary conditions we first write down the total Bromwich potentials for each of the three media. We let

$$\begin{aligned} U(\boldsymbol{r}, \omega) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\,\mathrm{d}\beta \exp\{\mathrm{i}(\alpha x+\beta y)\} U(\alpha, \beta, z, \omega) \\ V(\boldsymbol{r}, \omega) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\,\mathrm{d}\beta \exp\{\mathrm{i}(\alpha x+\beta y)\} V(\alpha, \beta, z, \omega). \end{aligned} \tag{4.1}$$

Then

$$\begin{aligned} U_i^{\text{tot}}(\alpha, \beta, z, \omega) &= \mathrm{e}^{\mathrm{i}\gamma z} U_i^{0}(\alpha, \beta, \omega) + U_i(\alpha, \beta, z, \omega) \\ V_i^{\text{tot}}(\alpha, \beta, z, \omega) &= \mathrm{e}^{\mathrm{i}\gamma z} V_i^{0}(\alpha, \beta, \omega) + V_i(\alpha, \beta, z, \omega). \end{aligned} \tag{4.2}$$

Here the subscript i refers to the different media. $U_i^0(\alpha, \beta, \omega)$ and $V_i^0(\alpha, \beta, \omega)$ are obtained from (3.20) and (3.21):

$$\begin{aligned} U_i^0(\alpha, \beta, \omega) &= \frac{\mathrm{i}\mu_0 e}{v_z}\,\frac{\beta v_x}{k^2-\omega^2\mu_0\epsilon'}\,\frac{1}{\gamma^2-\omega^2\mu_0\epsilon'} \\ V_i^0(\alpha, \beta, \omega) &= \frac{\mathrm{i}e}{\omega v_z}\,\frac{\alpha\gamma v_x+(\gamma^2-\omega^2\mu_0\epsilon')v_z}{\epsilon'(\alpha^2+\beta^2)+\epsilon''\gamma^2-\omega^2\mu_0\epsilon'\epsilon''}\,\frac{1}{\gamma^2-\omega^2\mu_0\epsilon'}. \end{aligned} \tag{4.3}$$

From (3.24) and (3.25) we let

$$\begin{aligned} p &= (\omega^2\mu_0\epsilon'-\alpha^2-\beta^2)^{1/2} \qquad && \operatorname{Im} p > 0 \\ q &= \left\{\omega^2\mu_0\epsilon' - \frac{\epsilon'}{\epsilon''}(\alpha^2+\beta^2)\right\}^{1/2} \qquad && \operatorname{Im} q > 0. \end{aligned} \tag{4.4}$$

Then we obtain

$$\begin{aligned} U_1(\alpha, \beta, z, \omega) &= \exp(-\mathrm{i}p_1 z) A_1(\alpha, \beta, \omega) \\ U_2(\alpha, \beta, z, \omega) &= \exp(-\mathrm{i}p_2 z) A_2(\alpha, \beta, \omega) + \exp(\mathrm{i}p_2 z) B_2(\alpha, \beta, \omega) \\ U_3(\alpha, \beta, z, \omega) &= \exp(\mathrm{i}p_3 z) B_3(\alpha, \beta, \omega). \end{aligned} \tag{4.5}$$

The radiation condition has been used to limit the form of the solution in media 1 and 3. Similarly

$$\begin{aligned} V_1(\alpha, \beta, z, \omega) &= \exp(-\mathrm{i}q_1 z) F_1(\alpha, \beta, \omega) \\ V_2(\alpha, \beta, z, \omega) &= \exp(-\mathrm{i}q_2 z) F_2(\alpha, \beta, \omega) + \exp(\mathrm{i}q_2 z) G_2(\alpha, \beta, \omega) \\ V_3(\alpha, \beta, z, \omega) &= \exp(\mathrm{i}q_3 z) G_3(\alpha, \beta, \omega). \end{aligned} \tag{4.6}$$

Imposing the boundary conditions, we obtain two sets of simultaneous linear equations:

$$
\begin{aligned}
\exp(-i\gamma a)U_1^0+\exp(ip_1 a)A_1 &= \exp(-i\gamma a)U_2^0+\exp(ip_2 a)A_2+\exp(-ip_2 a)B_2 \\
i\gamma\exp(-i\gamma a)U_1^0-ip_1\exp(ip_1 a)A_1 &= i\gamma\exp(-i\gamma a)U_2^0-ip_2\exp(ip_2 a)A_2 \\
&\quad +ip_2\exp(-ip_2 a)B_2 \\
\exp(i\gamma a)U_3^0+\exp(ip_3 a)B_3 &= \exp(i\gamma a)U_2^0+\exp(-ip_2 a)A_2+\exp(ip_2 a)B_2 \\
i\gamma\exp(i\gamma a)U_3^0+ip_3\exp(ip_3 a)B_3 &= i\gamma\exp(i\gamma a)U_2^0-ip_2\exp(-ip_2 a)A_2 \\
&\quad +ip_2\exp(ip_2 a)B_2
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
\epsilon_1\exp(-i\gamma a)V_1^0+\epsilon_1\exp(iq_1 a)F_1 &= \epsilon'\exp(-i\gamma a)V_2^0+\epsilon'\exp(iq_2 a)F_2+\epsilon'\exp(-iq_2 a)G_2 \\
i\gamma\exp(-i\gamma a)V_1^0-iq_1\exp(iq_1 a)F_1 &= i\gamma\exp(-i\gamma a)V_2^0-iq_2\exp(iq_2 a)F_2 \\
&\quad +iq_2\exp(-iq_2 a)G_2 \\
\epsilon_3\exp(i\gamma a)V_3^0+\epsilon_3\exp(iq_3 a)G_3 &= \epsilon'\exp(i\gamma a)V_2^0+\epsilon'\exp(-iq_2 a)F_2+\epsilon'\exp(iq_2 a)G_2 \\
i\gamma\exp(i\gamma a)V_3^0+iq_3\exp(iq_3 a)G_3 &= i\gamma\exp(i\gamma a)V_2^0-iq_2\exp(-iq_2 a)F_2 \\
&\quad +iq_2\exp(iq_2 a)G_2.
\end{aligned}
\tag{4.8}
$$

We are mainly interested in the secondary wave in medium 1. Therefore we solve for the amplitudes A_1 and F_1. Our task consists in evaluating four 4×4 determinants. However, all of them are special cases of the following one:

$$
\begin{aligned}
&\begin{vmatrix}
P & \epsilon'\exp(iq_2 a) & \epsilon'\exp(-iq_2 a) & 0 \\
Q & -iq_2\exp(iq_2 a) & iq_2\exp(-iq_2 a) & 0 \\
R & \epsilon'\exp(-iq_2 a) & \epsilon'\exp(iq_2 a) & -\epsilon_3\exp(iq_3 a) \\
S & -iq_2\exp(-iq_2 a) & iq_2\exp(iq_2 a) & -iq_3\exp(iq_3 a)
\end{vmatrix} \\
&= -2\exp(iq_3 a)[q_2 P\{\epsilon' q_3\cos(2q_2 a)-i\epsilon_3 q_2\sin(2q_2 a)\} \\
&\quad +i\epsilon' Q\{\epsilon_3 q_2\cos(2q_2 a)-i\epsilon' q_3\sin(2q_2 a)\}+i\epsilon' q_2(iq_3 R-\epsilon_3 S)].
\end{aligned}
\tag{4.9}
$$

We can now express the solutions in the following form:

$$
\begin{aligned}
A_1(\alpha,\beta,\omega) &= \exp(-ip_1 a)\frac{N(\alpha,\beta,\omega)}{D(\alpha,\beta,\omega)} \\
F_1(\alpha,\beta,\omega) &= \exp(-iq_1 a)\frac{N'(\alpha,\beta,\omega)}{D'(\alpha,\beta,\omega)}
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
D(\alpha,\beta,\omega) &= p_2(p_1+p_3)\cos(2p_2 a)-i(p_1 p_3+p_2^2)\sin(2p_2 a) \\
N(\alpha,\beta,\omega) &= \exp(-i\gamma a)\{p_2(\gamma-p_3)\cos(2p_2 a)-i(\gamma p_3-p_2^2)\sin(2p_2 a)\}(U_1^0-U_2^0) \\
&\quad +\exp(i\gamma a)p_2(\gamma-p_3)(U_2^0-U_3^0) \\
D'(\alpha,\beta,\omega) &= \epsilon' q_2(\epsilon_3 q_1+\epsilon_1 q_3)\cos(2q_2 a)-i(\epsilon'^2 q_1 q_3+\epsilon_1\epsilon_3 q_2^2)\sin(2q_2 a) \\
N'(\alpha,\beta,\omega) &= -\exp(-i\gamma a)q_2\{\epsilon' q_3\cos(2q_2 a)-i\epsilon_3 q_2\sin(2q_2 a)\}(\epsilon_1 V_1^0-\epsilon' V_2^0) \\
&\quad +\exp(-i\gamma a)\epsilon'\gamma\{\epsilon_3 q_2\cos(2q_2 a)-i\epsilon' q_3\sin(2q_2 a)\}(V_1^0-V_2^0) \\
&\quad -\exp(i\gamma a)\epsilon' q_2 q_3(\epsilon' V_2^0-\epsilon_3 V_3^0)+\exp(i\gamma a)\epsilon'\epsilon_3\gamma q_2(V_2^0-V_3^0).
\end{aligned}
\tag{4.11}
$$

5. The asymptotic field

We have obtained the solution of the problem in the form of double Fourier integrals:

$$\begin{aligned} U(\boldsymbol{r}, \omega) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta \exp[\mathrm{i}\{\alpha x+\beta y-p_1(z+a)\}]\frac{N(\alpha, \beta, \omega)}{D(\alpha, \beta, \omega)} \\ V(\boldsymbol{r}, \omega) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta \exp[\mathrm{i}\{\alpha x+\beta y-q_1(z+a)\}]\frac{N'(\alpha, \beta, \omega)}{D'(\alpha, \beta, \omega)}. \end{aligned} \tag{5.1}$$

An exact evaluation of the integrals is very difficult. However, for our purpose it is only necessary to derive the asymptotic form in the limit of large values of x, y and z. Firstly, it is convenient to introduce polar coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z' &= -(z+a) = r\cos\theta. \end{aligned} \tag{5.2}$$

This represents a left-handed coordinate system with the origin at $(0, 0, -a)$ and the polar axis along the negative z axis. Furthermore, in medium 1 we have

$$\begin{aligned} p_1 = q_1 &= (k_1{}^2-\alpha^2-\beta^2)^{1/2} \\ k_1 &= \omega(\mu_0\epsilon_1)^{1/2}. \end{aligned} \tag{5.3}$$

Then it can be shown that for large r we have the asymptotic relation (Born and Wolf 1959)

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{d}\beta\, f(\alpha, \beta)\exp[\mathrm{i}\{\alpha x+\beta y+(k_1{}^2-\alpha^2-\beta^2)^{1/2}z'\}] \\ &= -2\pi \mathrm{i}k_1\cos\theta\, f(k_1\sin\theta\cos\phi, k_1\sin\theta\sin\phi)\frac{\exp(\mathrm{i}k_1 r)}{r} \end{aligned} \tag{5.4}$$

where $f(\alpha, \beta)$ is a slowly varying function. Applying (5.4) to (5.1), we obtain the asymptotic form of the Bromwich potentials:

$$\begin{aligned} U(\boldsymbol{r}, \omega) &= \frac{-\mathrm{i}}{2\pi}k_1\cos\theta\frac{N(k_1\sin\theta\cos\phi, k_1\sin\theta\sin\phi, \omega)}{D(k_1\sin\theta\cos\phi, k_1\sin\theta\sin\phi, \omega)}\frac{\exp(\mathrm{i}k_1 r)}{r} \\ V(\boldsymbol{r}, \omega) &= \frac{-\mathrm{i}}{2\pi}k_1\cos\theta\frac{N'(k_1\sin\theta\cos\phi, k_1\sin\theta\sin\phi, \omega)}{D'(k_1\sin\theta\cos\phi, k_1\sin\theta\sin\phi, \omega)}\frac{\exp(\mathrm{i}k_1 r)}{r}. \end{aligned} \tag{5.5}$$

These represent a spherical wave diverging from the neighbourhood of the origin.

Actually, the asymptotic expressions (5.5) are valid only when the particle velocity is below the threshold of the generation of Čerenkov radiation. The reason is that the derivation of (5.4) is based on the method of steepest descent. At some stage of the calculation we have to deform the contour of integration into a path of steepest descent through a saddle point. In this deformation process the contour may sweep over poles of $f(\alpha, \beta)$ whose locations vary with the velocity of the incident particle. These poles yield additional contributions to (5.5). It can be shown that the condition for the inclusion of the contributions from these poles is precisely the same as that for the emission of Čerenkov radiation, namely the surpassing of the phase velocity of light in the media by the particle velocity. Thus the contributions from these poles represent Čerenkov radiation. However, the appearance of Čerenkov radiation in a transition radiation measurement is undesirable, since it complicates the situation. We should naturally try to keep the particle velocity below the threshold. Therefore we shall not go into a discussion of Čerenkov radiation. The reader who is interested in this aspect of the calculation is referred to the literature (Garibian 1958).

The radiation field is obtained by substituting (5.5) into (3.18):

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}, \omega) &= \frac{\mathrm{i}}{2\pi} k_1^2 \cos\theta \left\{\omega \frac{N}{D} \boldsymbol{n} \times \boldsymbol{e}_z + k_1 \frac{N'}{D'} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{e}_z)\right\} \frac{\exp(\mathrm{i}k_1 r)}{r} \\
\boldsymbol{B}(\boldsymbol{r}, \omega) &= \frac{\mathrm{i}}{2\pi} k_1^2 \cos\theta \left\{k_1 \frac{N}{D} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{e}_z) - \omega\mu_0\epsilon_1 \frac{N'}{D'} \boldsymbol{n} \times \boldsymbol{e}_z\right\} \frac{\exp(\mathrm{i}k_1 r)}{r}.
\end{aligned}
\tag{5.6}
$$

Here $\boldsymbol{n}$ is a unit vector along the direction of observation. In our left-handed coordinate system it is given by

$$
\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{5.7}
$$

We note that there is no need to include the primary field in (5.6) since, at a velocity below the threshold for the emission of Čerenkov radiation, the frequency component of the field of a uniformly moving charged particle falls off exponentially away from the source, and does not contribute to the radiation field.

The radiation field is polarized along two directions. From (5.6) we see that the transverse electric component is polarized along $\boldsymbol{n} \times \boldsymbol{e}_z$ and the transverse magnetic component along $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{e}_z)$. These two directions are orthogonal. Moreover, they lie in a plane orthogonal to the direction of observation $\boldsymbol{n}$. This polarization property is of great importance in the detection of transition radiation. It helps us to distinguish transition radiation from radiations of other origins.

From the radiation field (5.6) we can construct the Poynting vector $\boldsymbol{\Pi}(\boldsymbol{r}, \omega)$:

$$
\boldsymbol{\Pi}(\boldsymbol{r}, \omega) = \boldsymbol{E}(\boldsymbol{r}, \omega) \times \boldsymbol{H}(\boldsymbol{r}, \omega)^* = \left(\frac{\epsilon_1}{\mu_0}\right)^{1/2} |\boldsymbol{E}(\boldsymbol{r}, \omega)|^2 \boldsymbol{n} \tag{5.8}
$$

such that the total energy radiated per unit solid angle along a given direction from $t = -\infty$ to $t = \infty$ is given by

$$
\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{r^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \Pi(\boldsymbol{r}, \omega), \qquad r \to \infty. \tag{5.9}
$$

Thus at a given frequency ω the angular distribution of the radiation is given by

$$
\frac{\mathrm{d}^2 W}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{r^2}{2\pi} \Pi(\boldsymbol{r}, \omega). \tag{5.10}
$$

The angular distributions of the transverse electric and the transverse magnetic waves, taken individually, are

$$
\begin{aligned}
\frac{\mathrm{d}^2 W^{\mathrm{TE}}}{\mathrm{d}\Omega\,\mathrm{d}\omega} &= \frac{1}{(2\pi)^3} k_1^4 \omega^2 \left(\frac{\epsilon_1}{\mu_0}\right)^{1/2} \left|\frac{N}{D}\right|^2 \sin^2\theta\cos^2\theta \\
\frac{\mathrm{d}^2 W^{\mathrm{TM}}}{\mathrm{d}\Omega\,\mathrm{d}\omega} &= \frac{1}{(2\pi)^3} k_1^6 \left(\frac{\epsilon_1}{\mu_0}\right)^{1/2} \left|\frac{N'}{D'}\right|^2 \sin^2\theta\cos^2\theta.
\end{aligned}
\tag{5.11}
$$

Since the directions of polarization of the two waves are mutually perpendicular, the total radiation pattern is just the superposition of these two expressions.

In general, the energy radiated is proportional to the squares of the charge and the velocity of the particle. For particle velocities not too close to that of light the radiation pattern is relatively velocity independent. Moreover, since it depends only on a few parameters, namely the dielectric constants and the thickness $2a$, these can be determined by performing measurements on the radiation pattern. The experimental aspects of transition radiation are discussed at length in a review article of Frank (1966).

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